Suggested Solutions to:
Regular Exam, Fall 2014
Contract Theory, January 10, 2015

This version: February 4, 2015

Question 1: Adverse selection and optimal procurement

(a)

Explain in words what each one of the four constraints says and why it must be satisfied.

IR-bad: The IR-bad constraint says that the less able type of agent (with $\theta = \overline{\theta}$) must, at least weakly, prefer the contract aimed at her to the outside option. Choosing the contract yields the utility $\overline{t} - C\left(\overline{q}, \overline{\theta}\right)$ and the outside option yields the utility zero. If this condition was violated, the less able type of agent would not choose the contract that P wants her to choose, because the outside option yields a higher utility.

IR-good: The interpretation of the IR-good constraint is analogous to the one for IR-bad, but concerns the relatively able type (with $\theta = \underline{\theta}$).

IC-bad: The IC-bad constraint says that the less able type of agent must, at least weakly, prefer the contract aimed at her to the contract aimed at the relatively able agent. This condition must be satisfied for the less able agent to choose the contract P wants her to choose. P must ensure that this condition is satisfied because P cannot observe the agent's type directly and therefore is unable to instruct the agent to pick one of the two contracts: each agent type must have an incentive to voluntarily choose the one aimed at her.

IC-good: The interpretation of the IC-good constraint is analogous to the one for IC-bad, but concerns the relatively able type.

(b)

Prove that incentive compatibility and Spence-Mirrlees ($C_{q\theta} > 0$) imply monotonicity; that is, show that if the inequalities defining incentive compatibility hold and if the Spence-Mirrlees condition is satisfied, then the quantity offered to the $\underline{\theta}$-type agent is at least as large as the one offered to the $\overline{\theta}$-type agent.

Incentive compatibility means that IC-bad and IC-good hold. Adding these two inequalities yields:

$$\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + \left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] \geq \left[\overline{t} - C\left(\overline{q}, \underline{\theta}\right)\right] + \left[\underline{t} - C\left(\underline{q}, \overline{\theta}\right)\right].$$

The t's cancel out, so the above inequality simplifies to

$$C\left(\underline{q}, \overline{\theta}\right) - C\left(\overline{q}, \overline{\theta}\right) \geq C\left(\underline{q}, \underline{\theta}\right) - C\left(\overline{q}, \underline{\theta}\right).$$

Rewriting again on integral form yields

$$\int_{\overline{q}}^{\underline{q}} C_q\left(q, \overline{\theta}\right) dq \geq \int_{\overline{q}}^{\underline{q}} C_q\left(q, \underline{\theta}\right) dq.$$

Rewriting yet again, on double integral form, we obtain

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\overline{q}}^{\underline{q}} C_{q\theta}\left(q, \theta\right) dq d\theta \geq 0.$$

By $\overline{\theta} > \underline{\theta}$ and the Spence-Mirrlees property $C_{q\theta} > 0$, the last inequality implies $\underline{q} \geq \overline{q}$, which is what we were asked to prove.

(c)

The first best optimal quantities are defined by $S'\left(\underline{q}^{FB}\right) = C_q\left(\underline{q}^{FB}, \underline{\theta}\right)$ and $S'\left(\overline{q}^{FB}\right) = C_q\left(\overline{q}^{FB}, \overline{\theta}\right)$, respectively. Assume that the constraints (IR-good) and (IC-bad) are lax at the second-best optimum (so that they can be disregarded). Show that, at the second-best optimum, the good type's quantity is *not* distorted relative to the first best ($\underline{q}^{SB} = \underline{q}^{FB}$) and that the bad type's quantity is distorted downwards ($\overline{q}^{SB} < \overline{q}^{FB}$).

We are allowed to assume that (IR-good) and (IC-bad) are lax at the optimum. Given that, the problem can be written as: Choose $\left(\underline{t},\underline{q},\overline{t},\overline{q}\right)$ so as to maximize

$$\nu\left[S\left(\underline{q}\right)-\underline{t}\right]+(1-\nu)\left[S\left(\overline{q}\right)-\overline{t}\right],$$

subject to the following two constraints:

$$\overline{t}-C\left(\overline{q},\overline{\theta}\right)\geq 0, \tag{IR-bad}$$

$$\underline{t}-C\left(\underline{q},\underline{\theta}\right)\geq\overline{t}-C\left(\overline{q},\underline{\theta}\right). \tag{IC-good}$$

Claim: At the optimum of the problem above, both constraints must bind.

Proof of claim:

- Suppose, per contra, that we have an optimum and that IR-bad is lax. Then we can lower $\overline{t}$, while still satisfying both constraints (IC-good will actually be relaxed), thereby increasing the value of the objective function (for this is decreasing in $\overline{t}$). But that is impossible, since we started at an optimum. Hence IR-bad must bind at an optimum.
- Suppose, per contra, that we have an optimum and that IC-good is lax. Then we can lower $\underline{t}$, while still satisfying both constraints (IR-bad will not be affected), thereby increasing the value of the objective function (for this is decreasing in $\underline{t}$). But that is impossible, since we started at an optimum. Hence IC-good must bind at an optimum.

Given that both constraints bind, we can replace the inequalities with equalities and then solve for $\underline{t}$ and $\overline{t}$. Doing this we get:

$$\overline{t}-C\left(\overline{q},\overline{\theta}\right)=0\Rightarrow\overline{t}=C\left(\overline{q},\overline{\theta}\right)$$

and

$$\underline{t}-C\left(\underline{q},\underline{\theta}\right)=\overline{t}-C\left(\overline{q},\underline{\theta}\right)\Rightarrow$$

$$\begin{aligned}\underline{t}&=C\left(\underline{q},\underline{\theta}\right)+\overline{t}-C\left(\overline{q},\underline{\theta}\right)\\&=C\left(\underline{q},\underline{\theta}\right)+C\left(\overline{q},\overline{\theta}\right)-C\left(\overline{q},\underline{\theta}\right).\end{aligned}$$

Plugging these values of $\overline{t}$ and $\underline{t}$ into P's objective function yields

$$\begin{aligned}V&=\nu\left[S\left(\underline{q}\right)-\underline{t}\right]+(1-\nu)\left[S\left(\overline{q}\right)-\overline{t}\right]\\&=\nu\left[S\left(\underline{q}\right)-C\left(\underline{q},\underline{\theta}\right)-C\left(\overline{q},\overline{\theta}\right)+C\left(\overline{q},\underline{\theta}\right)\right]\\&\quad+(1-\nu)\left[S\left(\overline{q}\right)-C\left(\overline{q},\overline{\theta}\right)\right]\end{aligned}$$

P's problem is now to maximize the objective V above with respect to only two choice variables, $\underline{q}$ and $\overline{q}$.

The first-order condition with respect to $\underline{q}$ is:

$$\begin{aligned}\frac{\partial V}{\partial\underline{q}}&=\nu\left[S'\left(\underline{q}\right)-C_q\left(\underline{q},\underline{\theta}\right)\right]=0\\&\Rightarrow\ S'\left(\underline{q}^{SB}\right)=C_q\left(\underline{q}^{SB},\underline{\theta}\right).\end{aligned}$$

- This means that $\underline{q}^{SB}=\underline{q}^{FB}$, as we were asked to show.

The first-order condition with respect to $\overline{q}$ is:

$$\begin{aligned}\frac{\partial V}{\partial\overline{q}}&=\nu\left[-C_q\left(\overline{q},\overline{\theta}\right)+C_q\left(\overline{q},\underline{\theta}\right)\right]+(1-\nu)\left[S'\left(\overline{q}\right)-C_q\left(\overline{q},\overline{\theta}\right)\right]\\&=0\end{aligned}$$

or

$$(1-\nu)\,S'\left(\overline{q}\right)=(1-\nu)\,C_q\left(\overline{q},\overline{\theta}\right)+\nu\left[C_q\left(\overline{q},\overline{\theta}\right)-C_q\left(\overline{q},\underline{\theta}\right)\right]$$

or

$$S'\left(\overline{q}^{SB}\right)=C_q\left(\overline{q}^{SB},\underline{\theta}\right)+\frac{\nu}{1-\nu}\left[C_q\left(\overline{q}^{SB},\overline{\theta}\right)-C_q\left(\overline{q}^{SB},\underline{\theta}\right)\right].$$

- From the last equality we see that $\overline{q}^{SB}<\overline{q}^{FB}$ if and only if the last term on the right-hand side is strictly positive. We can write:

  $$\begin{aligned}\frac{\nu}{1-\nu}\left[C_q\left(\overline{q}^{SB},\overline{\theta}\right)-C_q\left(\overline{q}^{SB},\underline{\theta}\right)\right]&>0\Leftrightarrow\\C_q\left(\overline{q}^{SB},\overline{\theta}\right)-C_q\left(\overline{q}^{SB},\underline{\theta}\right)&>0\Leftrightarrow\\\int_{\underline{\theta}}^{\overline{\theta}}C_{q\theta}\left(\overline{q}^{SB},\theta\right)d\theta>0,&\end{aligned}$$

  which always holds due to the assumptions that $\overline{\theta}>\underline{\theta}$ and $C_{q\theta}>0$. This means that we indeed have $\overline{q}^{SB}<\overline{q}^{FB}$, as we were asked to show.

(d)

Explain the intuition for the results you were asked to show under (c). Also explain the nature of the trade-off that the principal faces.

The trade-off that the principal faces when solving the problem under asymmetric information is between, on the one hand, letting the agent types produce the efficient levels and, on the other hand, not to give away rents to the agent.

The reason why P cannot achieve both those goals is that he cannot observe A's type. In particular, if P offered contracts that involved full efficiency and no rent extraction, then the good type of agent would have an incentive to choose the contract aimed at the bad type of agent (so IC-good would be violated).

In order to make sure that IC-good is satisfied, P can do two things.

- First, he can make the bad type's contract less attractive in the eyes of the good type by asking the bad type to produce less (so a quantity below the efficient level). If doing that, P would need to pay less money to the bad type (to ensure that his IR constraint is satisfied), which makes the bad type's contract less attractive.
- Second, P can make the payment in the good type's contract larger, which again would lower the good type's incentive to choose the bad type's contract.
- P will find it optimal to do a little bit of both those things, thus distorting the bad type's quantity downwards and giving away some rents to the good type.

From the lecture slides:

Key to the results we have derived is that the good type is the one who gets, for any given q, both:

(i) the highest *marginal* utility [due to Spence-Mirrlees] and

(ii) the highest *total* utility.

Because of (ii), the principal's top priority is to make the good type choose his first-best quantity.

- That type can get a high utility level (relative to the outside option utility), which the principal then can grab a large part of.

If the principal were to take too much of the good type's utility, then that type would instead choose the bad type's bundle.

- To prevent this, the principal makes the bad type's bundle less attractive by lowering that type's quantity and payment.
- This way of separating the two types works because of (i): *The good type benefits less from a reduction in q than the bad type.*

Question 2: Moral hazard and insurance when there are three outcomes

(a)

Show that IR-H and IC bind at the optimum.

As stated in the question, P's problem is to to maximize its profits subject to IR-H and IC. The Lagrangian can be written as

$$\begin{aligned}\mathcal{L} &= \widehat{w} - \pi_{N1} h(u_N) - \pi_{S1} h(u_S) - \pi_{B1} h(u_B) \\ &\quad + \lambda \left[\pi_{N1} u_N + \pi_{S1} u_S + \pi_{B1} u_B - \psi - u(\widehat{w})\right] \\ &\quad + \mu \left[(\pi_{N1} - \pi_{N0}) u_N + (\pi_{S1} - \pi_{S0}) u_S + (\pi_{B1} - \pi_{B0}) u_B - \psi\right],\end{aligned}$$

where λ is the shadow price associated with IR-H and μ is the shadow price associated with IC. Next, consider the first-order condition w.r.t. P's three choice variables. The FOC w.r.t. u_S is:

$$\frac{\partial \mathcal{L}}{\partial u_S} = -\pi_{S1} h'(u_S) + \lambda \pi_{S1} + \mu (\pi_{S1} - \pi_{S0}) = 0. \tag{1}$$

The FOC w.r.t. u_B is:

$$\frac{\partial \mathcal{L}}{\partial u_B} = -\pi_{B1} h'(u_B) + \lambda \pi_{B1} + \mu (\pi_{B1} - \pi_{B0}) = 0. \tag{2}$$

The FOC w.r.t. u_N is:

$$\frac{\partial \mathcal{L}}{\partial u_N} = -\pi_{N1} h'(u_N) + \lambda \pi_{N1} + \mu (\pi_{N1} - \pi_{N0}) = 0. \tag{3}$$

Claim: At the second-best optimum we have

$$\lambda = \pi_{N1} h'(u_N) + \pi_{S1} h'(u_S) + \pi_{B1} h'(u_B) > 0,$$

which means that IR-H binds at the optimum.

Proof: Adding up the three FOCs, using the assumptions that $\pi_{N0} + \pi_{S0} + \pi_{B0} = 1$ and $\pi_{N1} + \pi_{S1} + \pi_{B1} = 1$, yields the equality. The inequality holds because $h' > 0$. □

Claim: At the second-best optimum we have $\mu > 0$, which means that IC binds at the optimum.

Proof. Suppose, per contra, that $\mu = 0$. Then (1) implies that $h'(u_S) = \lambda$. Similarly, $\mu = 0$ together with (2) and (3), respectively imply that $h'(u_B) = \lambda$, and $h'(u_N) = \lambda$. By assumption, the utility function u is strictly concave, which means that its inverse is strictly convex; hence h' is strictly increasing and we thus must have $u_N = u_S = u_B$ (i.e., full insurance). However, these equalities in the IC constraint above yields

$$[(\pi_{N1} - \pi_{N0}) + (\pi_{S1} - \pi_{S0}) + (\pi_{B1} - \pi_{B0})] u_N \geq \psi$$
$$\Leftrightarrow 0 \geq \psi, \quad \text{(IC)}$$

which is impossible. □

(b)

Show that, at the optimum, the relationship $u_S \geq u_B$ holds if, and only if, the following condition is satisfied:

$$\frac{\pi_{B0}}{\pi_{B1}} \geq \frac{\pi_{S0}}{\pi_{S1}} \quad \text{(MLRP)}$$

The FOC (1) yields

$$\pi_{S1} h'(u_S) = \lambda \pi_{S1} + \mu (\pi_{S1} - \pi_{S0})$$

or

$$h'(u_S) = \lambda + \mu \left(1 - \frac{\pi_{S0}}{\pi_{S1}}\right).$$

Similarly, the FOC (2) yields

$$h'(u_B) = \lambda + \mu \left(1 - \frac{\pi_{B0}}{\pi_{B1}}\right).$$

It follows from these equalities that

$$\begin{aligned} &h'(u_S) \geq \ h'(u_B) \\ &\quad \Leftrightarrow \lambda + \mu \left(1 - \frac{\pi_{S0}}{\pi_{S1}}\right) \geq \ \lambda + \mu \left(1 - \frac{\pi_{B0}}{\pi_{B1}}\right) \\ &\quad\quad\quad\quad\quad \Leftrightarrow \frac{\pi_{B0}}{\pi_{B1}} \geq \ \frac{\pi_{S0}}{\pi_{S1}}, \end{aligned}$$

where the last step uses $\mu > 0$. Moreover, since h' is strictly increasing, $h'(u_S) \geq \ h'(u_B)$ is equivalent to $u_S \geq u_B$. Thus, $u_S \geq u_B \Leftrightarrow \frac{\pi_{B0}}{\pi_{B1}} \geq \ \frac{\pi_{S0}}{\pi_{S1}}$, which we were asked to show.

(c)

One can show that also if, as assumed in the model, the condition (FOSD) is satisfied, the condition (MLRP) may be violated. This means that there exist parameter values for which, at the optimal contract, we have $u_B > u_S$; that is, A gets a *higher* utility after a big fire than after a small fire. Explain the intuition for why it can be optimal for P to design a contract with this feature.

In order to induce A to make an effort, P must underinsure A: If A got the same utility whether or not there was no fire, a small fire or a big fire, then why would he make a costly effort in order to avoid the fire? If we do have $u_B > u_S$, then that means that A is in some sense more underinsured in the case of a small fire than in the case of a big fire. Why would P want to choose such an insurance policy? To see this, note that if P wants to provide a strong incentive to make an effort, then he should primarily underinsure the outcome *for which choosing a high effort has a big (positive) impact on the probability of avoiding that outcome.* For example, suppose π_{S1} is smaller than π_{S0} (so making an effort helps avoiding a small fire) but we have $\pi_{B1} \approx \pi_{B0}$ (so making an effort has hardly no impact on the likelihood of a big fire). Then, if P wants to incentivize A to make an effort, underinsuring A in case of a small fire should be much more effective than underinsuring A in case of a big fire. Why is that? Well, there will be a big fire with (roughly) the same likelihood regardless of whether A makes an effort or not — so why should he look at the level of underinsurance after that kind of fire when deciding whether to make an effort? The condition (MLRP) ensures that making an effort has a sufficiently big impact on the big-fire likelihood, relative to the impact on the small-fire likelihood, to guarantee that $u_B \leq \ u_S$.

(d)

Show, formally, that there cannot be full insurance at the optimum. Also, provide verbal arguments for why, or why not, the optimal contract involves full insurance if P induces $e = 0$.

[Note that the first question was implicity answered above under (a). Here it is answered again and more explicitly, though.]

Proof that there cannot be full insurance at the optimum: Suppose, per contra, that we indeed had full insurance at the optimum, that is, that $u_N = u_S = u_B$. Using these equalities in (IC) yields

$$[\pi_{N1} - \pi_{N0} + \pi_{S1} - \pi_{S0} + \pi_{B1} - \pi_{B0}] u_N \geq \psi. \quad \text{(IC)}$$

But, by assumption, the probabilities for a given effort level add up to one: $\pi_{N0} + \pi_{S0} + \pi_{B0} = 1$ and $\pi_{N1} + \pi_{S1} + \pi_{B1} = 1$. Therefore the above inequality simplifies to $0 \geq \psi$, which contradicts the assumption that the effort cost is strictly positive. It follows that there cannot be full insurance at the optimum. □

Verbal arguments for why, or why not, the optimal contract involves full insurance if P induces $e = 0$: If P wants to induce $e = 0$, then the optimal contract will involve full insurance. The intuitive reason is that in this case P does not care about A's effort level: P has no ambition to try to incentivize A to make an effort (if A were to make an effort anyway, P can't be hurt by that). Therefore, the moral hazard feature of the model (i.e., that A's effort choice is not observable by P) does not matter. Stated slightly more technically, if P wants to induce $e = 0$ then there is no IC constraint that must be satisfied and which could give rise to underinsurance.